Univariate distributions :-

Sums :-

1. A random variable X can assume the values -1, 0, 1 with probabilities $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{6}$ respectively. Find the distribution function.

2. Evaluate the distⁿ function of the following distⁿ:- spectrum of the r.v X is $\{-1, 0, 2, 3\}$ with probabilities $P(X=-1)=\frac{1}{7}$, $P(X=0)=\frac{2}{7}$, $P(X=2)=\frac{3}{7}$, $P(X=3)=\frac{1}{7}$

3. Find the constant K such that the funcⁿ $f(x)$ given by

$$f_x(x) = K|x| \quad , \quad -2<x<2$$
$$= 0 \quad , \quad \text{elsewhere}$$

$K=\frac{1}{4}$

is a possible pdf and find its distⁿ function.

4. Find the value of K so that the function $f(x)$ given by

$$f(x) = Kx(2-x) \quad , \quad 0<x<2$$
$$= 0 \quad , \quad \text{elsewhere}$$

$\boxed{K=\frac{3}{4} \quad P(X>1)=\frac{1}{2}}$

is a pdf. Construct the dF and compute $P(X>1)$.

5. A radioactive source emits on the average 2·5 particles per second. calculate the probability that 2 or more particles will be emitted in an interval of 4 seconds. [ANS: $1-11e^{-10}$]

6. Find K so that the function $f(x)$ defined by

$$f(x) = K(x-[x]) \quad , \quad 1<x<2$$
$$= 0 \quad , \quad \text{elsewhere}$$

is a pdf. Also find the distⁿ function.

7. If X follows poisson (μ) distribution, find the distribution of $cX+d$ (c, d are positive constants)

8. X is uniformly distributed over the interval $(-2, 2)$. Find the distribution of the r.v. $\min\{X, 1\}$.

9. If X is uniformly distributed in the interval $(-1,1)$, find the distribution of $|X|$.

10. The r.v. X is distributed uniformly over the interval $(0,2)$. Find the d.F. of the larger root of the quadratic equation $t^2+2t-X=0$.

11. The pdf of X is $2xe^{-x^2}$, $x>0$ and zero otherwise. Find the pdf of $Y=X^2$.

12. If X is Normal $(0,1)$, find the distribution of e^X.

Transformation of random variable
[continuous case]

Let X be a r.v. having an absolutely continuous probability distⁿ given by its p.d.f $f(x)$, $-\infty < x < \infty$.
Let $Y = g(X)$ be another r.v. which is strictly monotonic such that $X = g^{-1}(Y)$ exists. Then the p.d.f. of Y will be

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f\{g^{-1}(y)\}\left|\frac{dx}{dy}\right|$$

Proof: **case-I**

Let $y = g(x)$ be monotonic increasing function of x, having one-to-one correspondence. Here $\frac{dy}{dx} > 0$

Now $X \le x \iff g(X) \le g(x)$

or $Y \le y$

i.e. the events $X \le x$ and $Y \le y$ are identical

$$\therefore P(X \le x) = P(Y \le y)$$

or $F(x) = F(y)$

Taking differentials, we have

$$dF(x) = dF(y) = dF \text{ (say)}$$

If p.d.f of Y be $f(y)$, then we have

$$f(y)\,dy = f(x)\,dx = f\{g^{-1}(y)\}\,dx$$

$$\text{or} \quad f(y) = f(x)\frac{dx}{dy} = f\{g^{-1}(y)\}\frac{dx}{dy} \quad \text{--- (1)}$$

case-II Let $y = g(x)$ be monotonic decreasing function i.e. $\frac{dy}{dx} < 0$.

Now, $X \le x \iff g(X) \ge g(x)$

or $Y \ge y$

$$\therefore P(Y \ge y) = P(X \le x)$$

$$1 - P(Y < y) = P(X \le x)$$

or, $1 - F(y) = F(x)$

Taking differential, we have

$$-dF(y) = dF(x)$$

$$-f(y)\,dy = f(x)\,dx = f\{g^{-1}(y)\}\,dx$$

$$\text{or, } f(y) = -f(x)\frac{dx}{dy} = -f\{g^{-1}(y)\}\frac{dx}{dy} \quad \text{--- (2)}$$

combining (1) and (2), we have

$$f(y) = f(x)\left|\frac{dx}{dy}\right| = f\{g^{-1}(y)\}\left|\frac{dx}{dy}\right|$$

Discrete case

Here the spectrum only changes, the corresponding probability masses remaining the same.

Let $y = g(x)$ be a continuous and strictly monotonic function so that a unique inverse function $x = g^{-1}(y)$ exists.

set $\quad y_i = g(x_i) \quad$ --- (1)

Since the transformation has a unique inverse

$$(X = x_i) \Leftrightarrow g(X) = g(x_i)$$

$$\text{or } Y = y_i$$

Hence the spectrum of Y consists of the pts y_i given by (1).

and $P(X = x_i) = P(Y = y_i)$

or, $p_{x_i} = p_{y_i}$

Important discrete distribution

1. Binomial (n,p)-distⁿ:-

A discrete distribution of a random variable X is said to be binomial if the spectrum is the set $\{0,1,2,\cdots n\}$ ie $X = x_i = i \quad i = 0,1,2,\cdots n$ and probability mass function (pmf) is given by

$$f_i = P(X = x_i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0,1,2,\cdots n$$

where $n\,(n \in N)$ and $p\,(0<p<1)$ are two parameters of the distribution.

This distribution is known as binomial (n,p)-distribution

Here, $\sum_{i=0}^{n} f_i = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \{p+(1-p)\}^n = 1.$

The number of success X in a Bernoullian sequence of n independent trials follows binomial distribution with parameters n and p. p is the probability of success in each trial.

2. Poisson distribution:

The prob. mass. function (p.m.f) of the distⁿ is given by $P_x = \dfrac{e^{-\mu}\mu^x}{x!}, \quad x = 0,1,2,\cdots$

where $\mu\,(>0)$ is the only parameter of the distⁿ.

Note: (i) $P_x \geq 0$ and $\displaystyle\sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^x}{x!} = e^{-\mu}\left(1+\mu+\frac{\mu^2}{2!}+\frac{\mu^3}{3!}+\cdots\right)$

$$= e^{-\mu} e^{\mu} = 1$$

(ii) Following are examples of poisson variate.

1) Number of errors in type per page in a book.
2) Number of defective articles per box of 100 articles
3) Number of telephone calls in a booth in an interval of time
4) Number of car passing by in a crossing of a busy road for a given interval of time.

6. Uniform distribution :-

The probability mass function of discrete uniform distribution is given by

$$f_i = P(X = x_i) = \frac{1}{n} \quad , \; i = 1, 2, \cdots n$$

The spectrum of the random variable X is $X = x_1, x_2, \cdots x_n$

Here, there is n step points $x_1, x_2, \cdots x_n$ with equal probability. That is why it is called uniform.

The basic need for pmf $f_i > 0$ and $\sum_{i=1}^{n} f_i = 1$ satisfied.

If $x_m \leq x \leq x_{m+1}$, $m < n$, then

$$P(X \leq x) = \frac{m}{n} \quad \text{and} \quad P(X \leq x_n) = 1$$

The distribution function is given by

$$F(x) = m/n \quad , \; x_m \leq x \leq x_{m+1} \; , \; m < n$$
$$= 1 \quad , \; x = x_n .$$

Important continuous distribution

1. Uniform or Rectangular distribution

A continuous random variable X is said to have uniform distribution in the interval $a<X<b$ if the probability density function is defined by

$$f(x) = \frac{1}{b-a}, \quad a<x<b$$
$$= 0, \quad \text{elsewhere}$$

(i) We have, $f(x) \geq 0$ as $b>a$

and $\int_a^b f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[x\right]_a^b = \frac{b-a}{b-a} = 1.$

(ii) The distribution function $F(x)$ is given by

$$F(x) = P(-\infty < X \leq x)$$

Here, $F(x) = 0$ when $x<a$

For $a \leq X \leq b$, $F(x) = P(-\infty < X \leq a) + P(a \leq X \leq x)$

$$= 0 + \int_a^x \frac{1}{b-a}\,dx = \frac{x-a}{b-a}$$

For $x>b$, $F(x) = P(-\infty < X < a) + P(a \leq x \leq b) + P(b < X \leq x)$

$$= 0 + \int_a^b \frac{dx}{b-a} + 0 = 1.$$

$$\therefore F(x) = 0 \quad \text{when } x<a$$
$$= \frac{x-a}{b-a} \quad \text{when } a \leq x \leq b$$
$$= 1 \quad \text{when } x>b$$

(iii) A point X is chosen at random in the interval $a \leq X \leq b$ in such a way that the probability that it lies in any sub interval is proportional to the length of the sub interval. Then, we can show that X is uniformly distributed over the interval (a,b) i.e, The random point X has a uniform distribution in The given interval.

2. Normal distribution:

The probability density function (pdf) of this distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-m)^2}, \quad -\infty < x < \infty$$

where m and $\sigma (>0)$ be two parameters of this distn.

In this case, we say that the random variable X follows $N(m, \sigma)$-distribution

(i) From this pdf, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} dx \qquad \text{let } \frac{x-m}{\sqrt{2}\sigma} = z$$

then $$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2} dz = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1.$$

(ii) The distribution function $F(x)$ is given by

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-m)^2}{2\sigma^2}} dx.$$

3. Standard normal distribution.

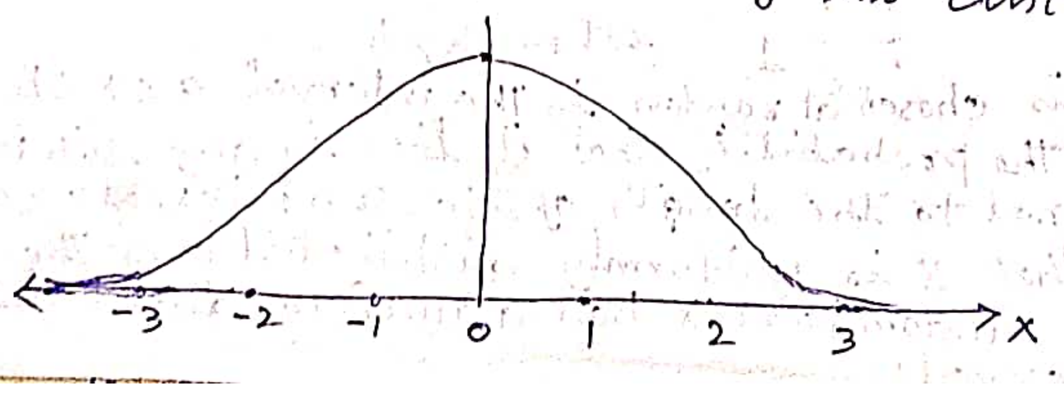
In normal distribution, we take $m=0$, $\sigma=1$, then the distribution follows $N(0,1)$.

The pdf of this distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad -\infty < x < \infty$$

The distribution function is $F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx$

The probability density curve of this distribution is



3. Exponential distribution:

The probability density function of this dist^n is given by

$$f(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \quad , \quad x > 0, \ \lambda > 0$$
$$= 0 \quad , \quad \text{elsewhere}$$

where $\lambda > 0$ is the only parameter of the distribution.

we see that (i) $f(x) \geq 0$ for all x and

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\,dx = 1$

Th: Show that poisson distribution as a limiting case of Binomial distribution.

proof: For binomial distribution, we have

$$f_x = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x=0,1,2,\cdots n$$

p is The probability of success in each trial.
Now, we take The number of trials n is large and p be very small. Then, np is moderate in magnitude.
Let $\mu = np$, then $p = \frac{\mu}{n}$

and $$P(X=x) = \frac{\lfloor n}{\lfloor x \lfloor n-x} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x}$$

Taking limit as $n \to \infty$ and $p \to 0$, we have

$$\lim_{n\to\infty} P(X=x) = \lim_{n\to\infty} \frac{\lfloor n}{\lfloor x \lfloor n-x} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x}$$

or, $$p_x = \lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{\lfloor x} \frac{\mu^x}{n^x} \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \frac{\mu^x}{\lfloor x} \lim_{n\to\infty} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) . \lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n \lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \frac{\mu^x}{\lfloor x} \cdot 1 \lim_{n\to\infty} \left[\left\{1+\left(-\frac{\mu}{n}\right)\right\}^{-\frac{n}{\mu}}\right]^{-\mu} \cdot 1 \quad \text{as } \lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-x} = 1$$

$$= \frac{\mu^x}{\lfloor x} e^{-\mu}, \quad \text{since } \lim_{x\to\infty}(1+x)^{\frac{1}{x}} = e$$

$$= \frac{e^{-\mu}\mu^x}{\lfloor x}, \quad x = 0,1,2,\cdots$$

which is The probability mass function of poisson distribution.

3. Negative Binomial distⁿ:

Let p be the prob. of success at each trial in an infinite sequence of independent Bernoullian trials. The random variable X denotes the number of failures preceeding the rth success ($r \geq 1$), then the distribution of X is given by the p.d.f

$$f_i = P(X=i) = \binom{i+r-1}{r-1} p^r (1-p)^i, \quad i = 0, 1, 2, \ldots$$

where, r and p $(0<p<1)$ be two parameters of the distⁿ.

4. Geometric distribution:

The r.v. X denote the number of failures preceeding the first success. Then the pdf of geometric distⁿ is given by

$$p_i = (1-p)^i p = q^i p, \quad i = 0, 1, 2, \ldots \quad p+q=1.$$

Note: Sometimes it is possible to develop such a formula even for a transformation which is not one-to-one.

Let $Y = X^2$ where X is a continuous r.v.

Here the transformation $y = x^2$ is not one-to-one.

We have, $F_Y(y) = P(Y \le y) = P(X^2 \le y)$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\therefore f(y) = F_Y'(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right]$$

$$= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}, \quad \text{if } y > 0$$

$$= 0 \qquad \text{if } y \le 0$$